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Estimation the parameters of the life of the new XLindley distribution under accelerated adaptive type-II progressively hybrid censored data with applications

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Abstract

In this paper, the parameters and corresponding lifetime parameters of the New X-Lindley distribution (NX-LD) are estimated. Collecting complete lifetime data from a population using highly reliable items is costly and time-consuming. An adaptive Type-II progressively hybrid censoring scheme under a constant-stress accelerated life test (ALT) is used to balance the optimal test time and the number of failed items. Under the constant-stress ALT model, a variety of statistical methods are developed to estimate the parameters of the lifetime distribution. These methods include maximum-likelihood and Bayesian point estimates. Asymptotic maximum-likelihood, bootstrap confidence intervals, and equal two-sided credible intervals are also formulated. These methods aim to obtain efficient estimators for the distribution parameter, acceleration factor, reliability, and failure rate indices. The estimators are assessed in terms of mean squared error, coverage percentage, and interval length through a Monte Carlo simulation study. The usefulness of the accelerated adaptive censoring scheme is demonstrated with a real-life application in the field of engineering, the voltage and failure time data, as well as a simulated data example to illustrate its practicality.

Keywords: Adaptive type-II progressively hybrid censored data; Bayesian estimation; Constant-partially accelerated life tests; Classical estimation; New X-Lindley distribution.

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1. Introduction

ALT is a technique applied to population items to accelerate the failure time of those items, such as high voltage, temp, vibration, or humidity. The main objective of ALT is to simulate the conditions of the real world, but in a faster way. Manufacturers used this technique to collect information on identifying weaknesses and how their products fail to make improvements. In literature, several types of ALTs, such as Constant Stress ALT, Step-Stress ALT, Progressive Stress ALT, Cyclic Stress ALT, or Random Vibration ALT, are discussed (Nelson, 1990). For example, in the constant stress ALT, applying a constant stress level makes test planning and execution relatively straightforward. It also enables accurate predictions of product lifespan because data analysis and modeling are simplified under constant stress levels. In this article, we applied the partially constant-stress ALT, which uses multiple stress levels; in other words, some items are tested under use conditions, and others are tested under stress conditions. Different works were presented in the context of partially constant-stress ALT, (Ismail and Tamimi, 2017, Algarni et al., 2019, Almalki et al., 2022 & Bakoban et al., 2024).

In life testing experiments, both the type-I and type-II censoring schemes (CS) are commonly used. The type-I CS fixes the test time but allows the failure time to vary. In a type-II CS, the test time varies, and the number of failures is fixed. These types of censoring don't allow units of the experiment to be removed at any time other than the endpoint of the experiment. The more general censoring scheme allows remove items before the endpoint, called the type-II progressive censoring scheme (PCS), see Balakrishnan and (Aggarwala, 2000). In type-II PCS, the number fails items and the corresponding items to be removed at each stage are proposed. It's a suitable approach in some cases, but other cases

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need to run the type-II PCS under constant test time η, which is known as the type-II progressive hybrid censoring scheme (type-II PHCS). In this scheme, an experiment is terminated at time point min(η , T_m), where T_m is the m -th failure time, see (Kundu and Joarder, 2006). The mechanism of type-II PHCS is more like type-I CS number of failure items may be small or zero. The problem with the efficiency of statistical inferences may require more information about the lives of individual items. Needing CS increase the efficiency of statistical testing by saving both the cost induced by failure and the total test time, (Ng et al., 2010) suggested an adaptive type-II PHCS. Testing under an adaptive type-II PHCS presents efficient testing and reduces testing time. Also, it provides more accurate estimates of lifetime parameters and reliability, as well as being adapted to different testing scenarios. For more details about an adaptive type-II PHCS, (Abd-Elmougod et al., 2015, Al-Essa et al., 2023 & Singh et al., 2024).

Suppose n independent items are randomly selected for testing. For each integer $m < n$, the ideal test time η and CS $r = (r_1, \; r_2, \; \dots, \; r_m)$, $n = m + \sum_{i=1}^{m} r_i$ are proposed in advance. During the experiment r_i items are randomly removed at the i -th failure until the ideal test time η is observed. The experimenter under an adaptive type-II PHCS completes the experiment without removing any items until the m -th failure time is observed. The remaining items are removed at the endpoint of the experiment. Therefore, if m -th failure time is shorter than η. An adaptive type-II PHCS is reduced to the usual type-II PCS. In the other case, if the experimental time passes time η and the number of observed failures has not reached m , i.e. $T_{k:m:\,n} < \eta < T_{k+1:m:n}$, $k = 1,2, \dots, m-1$. Then, CS $\boldsymbol{r}$ is corrected to be $\boldsymbol{R}$, where

$$\boldsymbol{R} = \begin{cases} R_i = r_i, \text{ if } i \le k \\ R_i = 0, \text{ if } k < i \le m-1 \\ R_m = n - m - \sum_{i=1}^{k} R_i \, , \text{ if } i = m \end{cases} \tag{1}$$

For more detail see (Ng et al., 2010). The schematic representation of an adaptive type-II PHCS is presented in Fig.1.

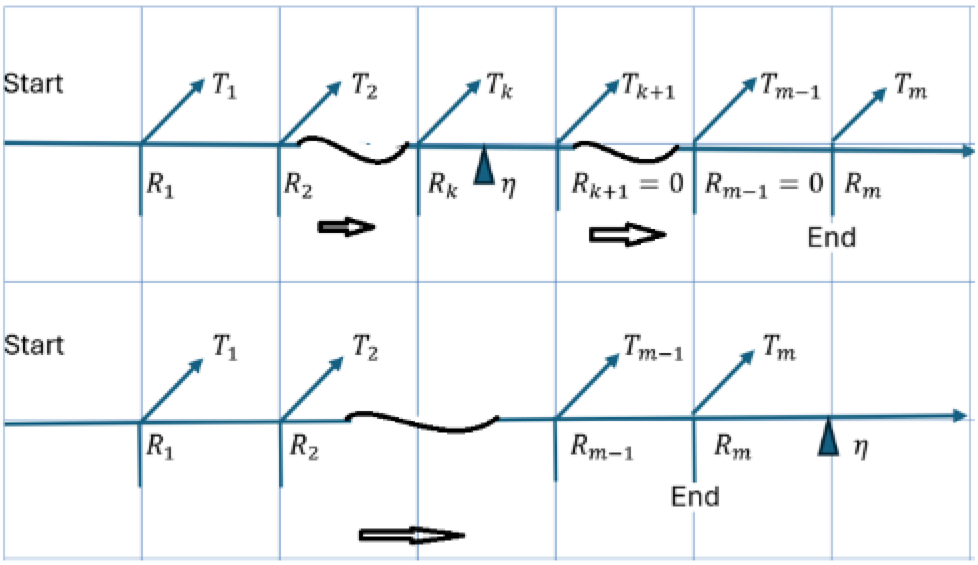

Fig. 1. The schematic representation of an adaptive type-II PHCS

Let $T_1, \; T_2, \; \dots, \; T_n$ be independent, identically distributed random variables with a probability density function (PDF) $f(t)$ and cumulative density function (CDF) $F(t)$. Given the ideal test time η , the CS $\boldsymbol{r}$ and the number of failure times needed for statistical inference m , an adaptive type-II HPCS is given by $t =\{ t_1, \; t_2, \; \dots, \; t_k, \; t_{k+1}, \; \dots, \; t_m \}$, where k is the number of items that fail before time η . The joint likelihood function under corrected CS $\boldsymbol{R}$ (1) is given by

$$f(t|\theta) = A \prod_{i=1}^{m} f(t_i) S(t_i)^{R_i}, \tag{2}$$

where

$$A = \prod_{i=1}^{m} \left[n - i + 1 - \sum_{i=1}^{min\{i-1,k\}} R_i \right]. \tag{3}$$

Although the monitoring effort and the complexity of experiment design with respect to the Type-II progressive hybrid censoring scheme, it offers important advantages. These include potentially lower overall testing costs, especially when units are expensive or time-consuming to observe, improved flexibility in handling withdrawals, and reduced experimental duration. However, less efficient use or longer test durations can result in simpler censoring schemes. From a management and operational perspective, an adaptive Type-II provides a balance between experimental efficiency and cost control. Therefore, the flexibility of adaptive Type-II is beneficial in reliability studies when testing time, labor, and maintenance costs are high. The problem of choosing a suitable censoring scheme should depend on the practical trade-off between implementation complexity and the economic value of reduced testing duration and improved inferential efficiency.

From a perspective of simplicity and flexibility, one-parameter distributions that can be used to model lifetime data are known as the Lindley distribution (LD). Early (Lindley, 1958) introduced LD under Bayesian approaches, and (Ghitany et al., 2008) used it for modeling reliability and lifetime data. Because of non-constant failure rates, LD is better than the exponential distribution at fitting real-world characteristics. In medical research and environmental studies, the Lindley distribution is widely applicable. It is characterized by a long right tail in the probability density function that decreases rapidly near zero. It also has applications in survival analysis and reliability modelling. In recent years, growing interest in this distribution has led to more in-depth investigation of its statistical properties and the development of new estimation techniques. Searching for a more flexible distribution is needed to analyse different types of lifetime data. Several one-parameter models include the Shanker distribution, the x-gamma distribution (Sen et al., 2016), the Ishita distribution (Shanker, 2017), the Zeghdoudi distribution (Messaadia and Zeghdoudi, 2018), and recently (Badr, 2025). A NE-LD, presented by (Khodja et al., 2023) as a special mixture formula from exponential and gamma distributions. Applications to failure data, including entropy and Fisher information for the new X-Lindley model, are formulated by (Husseiny et al., 2025).

The random variable T has an NX-LD if it has a cumulative distribution function (CDF) and probability density function (PDF) given by

$$F(t) = 1 - \left(\frac{1}{2}\beta t + 1\right) exp\{-\beta t\},\ t > 0,\ \beta > 0 \tag{4}$$

The main objective of this study is to estimate the parameters of the NX-LD and the corresponding life parameters, such as the reliability and hazard rate functions. Under a partially accelerated life test, data are collected to formulate a model describing the product's lifetime distribution under normal operating conditions. By applying a constant stress level to an adaptive type-II HPCS, the test is terminated when a predetermined number of failures occur within a short period of time. This yields reliable estimates of the model parameters and helps determine the product's warranty period, maintenance schedule, and overall reliability.

The rest of the paper is organized as. In Section 2, we introduce a description of the model and some basic assumptions. Section 3 deals with the derivation of the maximum likelihood estimators of the involved parameters, and the approximate confidence intervals (CIs) for the parameters are also presented. In Section 4, the two parametric bootstrap CIs are derived. In Section 5, the Bayesian approach is applied with the help of the MCMC method. In Section 6, Monte Carlo simulation results are presented. Data analysis is provided in Section 7. Section 8, the numerical discussion is reported. Finally, the paper concludes in section 9.

2. Model formulation and Assumptions

The mechanism of an adaptive type-II HPCS with respect to partially constant-stress ALT can be described as follows. Suppose N items are randomly selected from the population to be tested. From these items, n_1 items are tested under the use condition, and the remaining $n_2 = N - n_1$ under the accelerated condition. Also, let each of the total test time η, number of failure items m_j, and CSs $\mathbf{r}_j = (r_{1j},\ r_{2j},\ \dots,\ r_{m_j j})$, $j = 1,\ 2$ are prior proposed. When the experiment runs, at the first failure T_{1j} , remove from the $n_j - 1$ items r_{1j} randomly. At the second failure T_{2j} , randomly remove r_{2j} items from $n_j - 2$ - r_{1j} . The experiment is continual until the time η is observed. Supposed that, $T_{k_j} < \eta < T_{(k_j+1)}$ and $k_j = 1,2,\dots,m_j - 1$. So, the CSs $\mathbf{r}_j$ are corrected to $\mathbf{R}_j$ to satisfy $R_{k_j+1} = = R_{(m_j-1)} = 0$ and $R_{m_j} = n_j - m_j - \sum_{i=1}^{k_j} R_{ij}$ as given in (1). Under consideration, the random variable $T = \{T_{1j},\ T_{2j},\ \dots, T_{k_j j}, T_{(k_j+1)j},\ \dots,\ T_{m_j j}\}$ have the populations with PDFs and CDFs given by $f_j(.)$ and $F_j(.)$, $j = 1,2.$ The joint likelihood function from the observed data $t = \{t_{1j},\ t_{2j},\ \dots, t_{k_j j}, t_{(k_j+1)j},\ \dots,\ t_{m_j j}\}$ and CSs $\boldsymbol{R}_j$ give by

$$f(t|\theta) = \prod_{j=1}^{2} A_j \prod_{i=1}^{m_j} f_j(t_{ij}) S_j(t_{ij})^{R_{ij}}, \tag{5}$$

$$A_j = \prod_{i=1}^{m_j} \left[n_j - i + 1 - \sum_{i=1}^{min\{i-1,k_j\}} R_{ij} \right], \tag{6}$$

Remark 1: An adaptive Type-II PHCS is a general scheme that includes ordinary Type-II CS when $\eta \to 0$. Type-II PCS when $\eta \to \infty$ as a special case. Also, Complete data when $r = \underline{0}$ and $n = m$.

Suppose that the lifetime of items under use condition has NE-LD with CDF given by (4). The corresponding distribution under accelerated conditions can be obtained by putting the hazard rate function $h_2(t) = \theta h_1(t)$, where θ is an accelerated factor satisfy $\theta > 1$ and $h(.)$ is the hazard failure rate (HFR) function under use and accelerated conditions, respectively. Therefore, the NX-LD in two cases defines as

$$h(t) = \begin{cases} h_1(t) = \beta \frac{1+\beta t}{2+\beta t}, \\ h_2(t) = \beta\theta \frac{1+\beta t}{2+\beta t}, \end{cases}, \tag{7}$$

the corresponding $S(t)$, $F(t)$ and $f(t)$ in two cases are given by

$$S(t) = \begin{cases} S_1(t) = \left(\frac{1}{2}\beta t + 1\right) exp\{-\beta t\} \\ S_2(t) = \left(\frac{1}{2}\beta t + 1\right)^{\theta} exp\{-\beta\theta t\} \end{cases}, \tag{8}$$

$$F(t) = \begin{cases} F_1(t) = 1 - \left(\frac{1}{2}\beta t + 1\right) exp\{-\beta t\} \\ F_2(t) = 1 - \left(\frac{1}{2}\beta t + 1\right)^{\theta} exp\{-\beta\theta t\}, \end{cases} \tag{9}$$

and

$$f(t) = \begin{cases} f_1(t) = \frac{1}{2}\beta(\beta t + 1)\, exp\{-\beta t\} \\ f_2(t) = \frac{1}{2}\theta\beta(\beta t + 1)\left(\frac{1}{2}\beta t + 1\right)^{\theta-1} exp\{-\beta\theta t\} \end{cases}, t > 0, \beta > 0, \theta > 1. \tag{10}$$

3. Maximum Likelihood Estimation

The popular classical method for parameter estimation is maximum-likelihood estimation (MLE). In this method, we search for parameter values that maximize the likelihood function given the observed data. So, we discuss the point and interval ML estimate of the model parameters. The joint likelihood function (5) of an adaptive Type-II PHCS, $t = \{t_{1j}, t_{2j}, \ldots, t_{k_j j}, t_{(k_j+1)j}, \ldots, t_{m_j j}\}$, $j = 1, 2,$ with populations (4) and (7) is given

$$L(\beta,\theta|t) = C\beta^{m_1+m_2}\theta^{m_2} exp\left\{\sum_{i=1}^{m_1} log[\beta t_{i1} + 1] + \sum_{i=1}^{m_2} log[\beta t_{i2} + 1] - \sum_{i=1}^{m_2} log\left[\frac{1}{2}\beta t_{i2} + 1\right] \right.$$
$$\left. + \sum_{i=1}^{m_1} R_{i1}\, log\left[\frac{1}{2}\beta t_{i1} + 1\right] + \theta \sum_{i=1}^{m_2} (R_{i2} + 1)\, log\left[\frac{1}{2}\beta t_{i2} + 1\right] - \beta \sum_{i=1}^{m_1} (R_{i1} + 1) t_{i1} - \beta\theta \sum_{i=1}^{m_2} (R_{i2} + 1) t_{i2} \right\}, \tag{11}$$

where $C = \left(\frac{1}{2}\right)^{m_1+m_2} A_1 A_2$. Without the constant value C , the log-likelihood function is then given by

$$\ell(\beta,\theta|t) = (m_1 + m_2)\, log\,\beta + m_2\, log\,\theta + \sum_{i=1}^{m_1} log[\beta t_{i1} + 1] + \sum_{i=1}^{m_2} log[\beta t_{i2} + 1] - \sum_{i=1}^{m_2} log\left[\frac{1}{2}\beta t_{i2} + 1\right] +$$
$$\sum_{i=1}^{m_1} R_{i1}\, log\left[\frac{1}{2}\beta t_{i1} + 1\right] + \theta \sum_{i=1}^{m_2} (R_{i2} + 1)\, log\left[\frac{1}{2}\beta t_{i2} + 1\right] - \beta \sum_{i=1}^{m_1} (R_{i1} + 1) t_{i1} - \beta\theta \sum_{i=1}^{m_2} (R_{i2} + 1) t_{i2}. \tag{12}$$

3.1 MLEs

ML estimate of the parameters of a statistical model or any parameter of life given data, depends on the parameter value that maximizes the log-likelihood function as follows.

The likelihood equations are obtained from (12) by taking the first partial derivatives with respect to β and θ as

$$\frac{\partial \ell(\beta,\theta|t)}{\partial \beta}=\frac{m_1+m_2}{\beta}+\sum_{i=1}^{m_1}\frac{t_{i1}}{\beta t_{i1}+1}+\sum_{i=1}^{m_2}\frac{t_{i2}}{\beta t_{i2}+1}-\frac{1}{2}\sum_{i=1}^{m_2}\frac{t_{i2}}{\frac{1}{2}\beta t_{i2}+1}+\frac{1}{2}\sum_{i=1}^{m_1}\frac{R_{i1}t_{i1}}{\frac{1}{2}\beta t_{i1}+1}$$
$$+\frac{1}{2}\theta\sum_{i=1}^{m_2}\frac{(R_{i2}+1)t_{i2}}{\frac{1}{2}\beta t_{i2}+1}-\sum_{i=1}^{m_1}(R_{i1}+1)t_{i1}-\theta\sum_{i=1}^{m_2}(R_{i2}+1)t_{i2}=0, \tag{13}$$

and

$$\frac{\partial \ell(\beta,\theta|t)}{\partial \theta}=\frac{m_2}{\theta}+\sum_{i=1}^{m_2}(R_{i2}+1)\,log\left[\frac{1}{2}\beta t_{i2}+1\right]-\beta\sum_{i=1}^{m_2}(R_{i2}+1)t_{i2}=0. \tag{14}$$

The likelihood equation (14) reduces to

$$\theta(\beta)=\frac{m_2}{\beta\sum_{i=1}^{m_2}(R_{i2}+1)t_{i2}-\sum_{i=1}^{m_2}(R_{i2}+1)\,log\left[\frac{1}{2}\beta t_{i2}+1\right]} \tag{15}$$

By using the inequality $\log(1+x)<x$ It's clear that $\frac{1}{2}\beta t_{i2}-log\left[\frac{1}{2}\beta t_{i2}+1\right]>0$. Therefore, we can say that $\beta\sum_{i=1}^{m_2}(R_{i2}+1)t_{i2}-\sum_{i=1}^{m_2}(R_{i2}+1)\,log\left[\frac{1}{2}\beta t_{i2}+1\right]>0$. This show $\theta(\beta)$ is positive for all values of β.
Also from (15), the likelihood equation (13) is reduced to a single non-linear equation of β is given by

$$\frac{m_1+m_2}{\beta}+\sum_{i=1}^{m_1}\frac{t_{i1}}{\beta t_{i1}+1}+\sum_{i=1}^{m_2}\frac{t_{i2}}{\beta t_{i2}+1}-\frac{1}{2}\sum_{i=1}^{m_2}\frac{t_{i2}}{\frac{1}{2}\beta t_{i2}+1}+\frac{1}{2}\sum_{i=1}^{m_1}\frac{R_{i1}t_{i1}}{\frac{1}{2}\beta t_{i1}+1}-\sum_{i=1}^{m_1}(R_{i1}+1)t_{i1}$$
$$+\frac{m_2\left(\frac{1}{2}\sum_{i=1}^{m_2}\frac{(R_{i2}+1)t_{i2}}{\frac{1}{2}\beta t_{i2}+1}-\sum_{i=1}^{m_2}(R_{i2}+1)t_{i2}\right)}{\beta\sum_{i=1}^{m_2}(R_{i2}+1)t_{i2}-\sum_{i=1}^{m_2}(R_{i2}+1)\,log\left[\frac{1}{2}\beta t_{i2}+1\right]}=0 \tag{16}$$

We use the behavior of the score function $U(\beta)=\frac{\partial \ell(\beta,\theta|t)}{\partial \beta}$ in (13) to establish the existence and uniqueness of the estimator $\hat{\beta}$.

1) **Existence of the Solution:** From (13) $U(\beta)=\frac{\partial \ell(\beta,\theta|t)}{\partial \beta}|_{\hat{\theta}}$ is a continuous function for all values of $\beta>0$**.** The dominant terms in the function $U(\beta)$ describe as
   When $\beta\to 0^+$ all terms are finite, other than $\frac{m_1+m_2}{\beta}\to+\infty$, hence $\lim_{\beta\to 0^+}U(\beta)=+\infty$
   Next, as $\beta\to\infty$, all terms $\frac{t_{ij}}{\beta t_{ij}+1}\to 0$ and $\frac{t_{ij}}{\frac{1}{2}\beta t_{ij}+1}\to 0,\ j=1,2.$
   Consequently, $\lim_{\beta\to\infty}U(\beta)=-\sum_{i=1}^{m_1}(R_{i1}+1)t_{i1}<0$, Hence, the continuous function changes from positive to negative within its range, and the intermediate value theorem guarantees the existence of at least one root for the equation $U(\hat{\beta})=0$. Therefore, the maximum likelihood estimator of βexists.
2) **Uniqueness of the Solution:** Let us consider the derivative of the score function $U(\beta)$

$$\frac{dU(\beta)}{d\beta}|_{\hat{\theta}}=-\left(\frac{m_1+m_2}{\beta^2}\right)-\sum_{i=1}^{m_1}\frac{t^2{}_{i1}}{(\beta t_{i1}+1)^2}-\sum_{i=1}^{m_2}\frac{t^2{}_{i2}}{(\beta t_{i2}+1)^2}+\frac{1}{4}\sum_{i=1}^{m_2}\frac{t^2{}_{i2}}{\left(\frac{1}{2}\beta t_{i2}+1\right)^2}-\frac{1}{4}\sum_{i=1}^{m_1}\frac{R_{i1}t^2{}_{i1}}{\left(\frac{1}{2}\beta t_{i1}+1\right)^2}$$
$$-\frac{m_2\left(\frac{1}{4}\sum_{i=1}^{m_2}\frac{(R_{i2}+1)t^2{}_{i2}}{\left(\frac{1}{2}\beta t_{i2}+1\right)^2}\right)}{\beta\sum_{i=1}^{m_2}(R_{i2}+1)t_{i2}-\sum_{i=1}^{m_2}(R_{i2}+1)\,log\left[\frac{1}{2}\beta t_{i2}+1\right]}.$$

Due to $\beta\sum_{i=1}^{m_2}(R_{i2}+1)t_{i2}-\sum_{i=1}^{m_2}(R_{i2}+1)\,log\left[\frac{1}{2}\beta t_{i2}+1\right]>0$, hence $\frac{dU(\beta)}{d\beta}|_{\hat{\theta}}<0$ for all values β . Therefore, $\frac{dU(\beta)}{d\beta}|_{\hat{\theta}}$ is a concave function of β. Hence, the maximum likelihood estimator $\hat{\beta}$ is unique.

The following theorem is used to obtain the MLE of β by solving the single nonlinear equation (16)

Theorem: The fixed-point iteration is used to construct the ML estimate of the parameters β which defined by

$$\beta = \frac{-(m_1+m_2)}{g(\beta)}, \tag{17}$$

where

$$g(\beta) = \sum_{i=1}^{m_1} \frac{t_{i1}}{\beta t_{i1}+1} + \sum_{i=1}^{m_2} \frac{t_{i2}}{\beta t_{i2}+1} - \frac{1}{2}\sum_{i=1}^{m_2} \frac{t_{i2}}{\frac{1}{2}\beta t_{i2}+1} + \frac{1}{2}\sum_{i=1}^{m_1} \frac{R_{i1}t_{i1}}{\frac{1}{2}\beta t_{i1}+1} - \sum_{i=1}^{m_1}(R_{i1}+1)t_{i1}$$
$$+ \frac{m_2\left(\frac{1}{2}\sum_{i=1}^{m_2}\frac{(R_{i2}+1)t_{i2}}{\frac{1}{2}\beta t_{i2}+1} - \sum_{i=1}^{m_2}(R_{i2}+1)t_{i2}\right)}{\beta\sum_{i=1}^{m_2}(R_{i2}+1)t_{i2} - \sum_{i=1}^{m_2}(R_{i2}+1)\log\left[\frac{1}{2}\beta t_{i2}+1\right]}. \tag{18}$$

Remark 2: The profile log-likelihood from (12) provides a good initial value for iterative methods like fixed point or Newton-Raphson which are formulated by

$$h(\beta|t) = m_1 \log\beta + m_2 \log\left[\frac{\beta m_2}{\beta\sum_{i=1}^{m_2}(R_{i2}+1)t_{i2} - \sum_{i=1}^{m_2}(R_{i2}+1)\log\left[\frac{1}{2}\beta t_{i2}+1\right]}\right]$$
$$+\sum_{i=1}^{m_1}\log[\beta t_{i1}+1] + \sum_{i=1}^{m_2}\log[\beta t_{i2}+1] - \sum_{i=1}^{m_2}\log\left[\frac{1}{2}\beta t_{i2}+1\right]$$
$$+\sum_{i=1}^{m_1} R_{i1}\log\left[\frac{1}{2}\beta t_{i1}+1\right] - \beta\sum_{i=1}^{m_1}(R_{i1}+1)t_{i1} + m_2 \tag{19}$$

The ML estimate of the parameters of life reliability and the hazard failure rate functions of the system are given by

$$\hat{S}(t) = \left(\frac{1}{2}\hat{\beta}t+1\right)\exp\{-\hat{\beta}t\}, \tag{20}$$

and

$$\hat{h}(t) = \hat{\beta}\,\frac{1+\hat{\beta}t}{2+\hat{\beta}t}. \tag{21}$$

3.2 Approximate interval estimation

The asymptotic normality of maximum likelihood estimators (MLEs) is used to construct confidence intervals for model parameters. The Fisher information matrix, which is defined as the negative expectation of the second derivative of the log-likelihood, is utilized. In some cases, especially when there are many model parameters, it is replaced by an approximate information matrix (AIM) defined by

$$AIM_0 = AIM(\hat{\beta},\hat{\theta}) = \begin{pmatrix} -\frac{\partial^2\ell(\beta,\theta|t)}{\partial\beta 2} & -\frac{\partial^2\ell(\beta,\theta|t)}{\partial\beta\partial\theta} \\ -\frac{\partial^2\ell(\beta,\theta|t)}{\partial\theta\partial\beta} & -\frac{\partial^2\ell(\beta,\theta|t)}{\partial\theta^2} \end{pmatrix}_{(\hat{\beta},\hat{\theta})} \tag{22}$$

From the log-likelihood function in (15), we have

$$\frac{\partial^2\ell(\beta,\theta|t)}{\partial\beta^2} = \frac{-(m_1+m_2)}{\beta^2} - \sum_{i=1}^{m_1}\frac{t_{i1}^2}{(\beta t_{i1}+1)^2} - \sum_{i=1}^{m_2}\frac{t_{i2}^2}{(\beta t_{i2}+1)^2} + \frac{1}{4}\sum_{i=1}^{m_2}\frac{t_{i2}^2}{\left(\frac{1}{2}\beta t_{i2}+1\right)^2}$$
$$-\frac{1}{4}\sum_{i=1}^{m_1}\frac{R_{i1}t_{i1}^2}{\left(\frac{1}{2}\beta t_{i1}+1\right)^2} - \frac{1}{4}\theta\sum_{i=1}^{m_2}\frac{(R_{i2}+1)t_{i2}^2}{\left(\frac{1}{2}\beta t_{i2}+1\right)^2}, \tag{23}$$

$$\frac{\partial^2\ell(\beta,\theta|t)}{\partial\theta^2} = -\frac{m_2}{\theta^2}, \tag{24}$$

and

$$\frac{\partial^2\ell(\beta,\theta|t)}{\partial\beta\partial\theta} = \frac{\partial^2\ell(\beta,\theta|t)}{\partial\theta\partial\beta} = \frac{1}{2}\sum_{i=1}^{m_2}\frac{(R_{i2}+1)t_{i2}}{\frac{1}{2}\beta t_{i2}+1} - \sum_{i=1}^{m_2}(R_{i2}+1)t_{i2}. \tag{25}$$

To construct the approximate confidence intervals, use the following steps

1) Compute the ML estimate of the parameter β and θ, say $\hat{\beta}$ and $\hat{\theta}$.

2) Compute the standard error SE = $\sqrt{AIM_0^{-1}}$, where AIM_0 is the approximate information matrix.
3) Choose a suitable confidence level (CL) (1 - 2 α), for example, when choose 95% C I $\alpha = 2.5\%$.
4) Find the quantile value of the standard normal Z_α using Z -table.
5) The approximate interval estimators are given by $\hat{\beta} \mp Z_\alpha$ SE 1 and $\hat{\theta} \mp Z_\alpha$ SE 2, where SE is the diagonal of AIM_0^{-1} .

4. Bootstrap Confidence Intervals

In statistical inference, the bootstrap technique is a versatile resampling method used to estimate the bias, variance, and confidence intervals of estimators. The parametric percentile bootstrap methods are employed, which use quantiles of the bootstrap distribution. This method, when traditional parametric assumptions are questionable, provides practical ways to quantify uncertainty in parameter estimates. For more details, see (Efron, 1982, Hall, 1988, Efron and Tibshirani 1993, & Davison and Hinkley, 1997). The following algorithm is used to formulate the bootstrap point and interval estimate of model parameters and the corresponding lifetime parameters.

Algorithm 1: Bootstrap confidence interval.

1) **Input:** n, n_j, m_j, η, and the CSs r_j, $j = 1,2$.
2) Compute the ML estimate $\hat{\beta}$ and $\hat{\theta}$ based on the original accelerated adaptive type-II PHC sample, $t = \{t_{1j}, t_{2j}, \ldots, t_{k_j j}, t_{(k_j+1)j}, \ldots, t_{m_j j}\}$.
3) Based on the ML estimate $\hat{\beta}$ and $\hat{\theta}$ generate from (9) by using inverse transformation method with algorithm described in Ng et al. [Ng et al.2010] the bootstrap random sample $t^* = \{t^*_{1j}, t^*_{2j}, \ldots, t^*_{k_j j}, t^*_{(k_j+1)j}, \ldots, t^*_{m_j j}\}$.
4) Based on t^* compute the ML estimate $\hat{\beta}^*$ and $\hat{\theta}^*$ and the corresponding $\hat{S}^*(t)$ and $\hat{h}(t)^*$ for given t.
5) Steps 3 and 4 are repeated N times.
6) Arrange in acceding order all vectors $\hat{\beta}^* = (\hat{\beta}^{1*}, \hat{\beta}^{2*}, \ldots, \hat{\beta}^{N*})$, $\hat{\theta}^* = (\hat{\theta}^{1*}, \hat{\theta}^{2*}, \ldots, \hat{\theta}^{N*})$, $\hat{S}^*(t) = (\hat{S}^{*1}, \hat{S}^{2*}, \ldots, \hat{S}^{N*})$ and $\hat{h}(t)^* = (\hat{h}^{1*}, \hat{h}^{2*}, \ldots, \hat{h}^{N*})$ to obtain

$$\hat{g}_l^* = \begin{cases} (\hat{\beta}^*_{(1)}, \hat{\beta}^*_{(2)}, \ldots, \hat{\beta}^*_{(N)}), l = 1 \\ (\hat{\theta}^*_{(1)}, \hat{\theta}^*_{(2)}, \ldots, \hat{\theta}^*_{(N)})\ l = 2 \\ (\hat{S}^*_{(1)}, \hat{S}^*_{(2)}, \ldots, \hat{S}^*_{(N)})\ l = 3 \\ (\hat{h}^{1*}_{(1)}, \hat{h}^*_{(2)}, \ldots, \hat{h}^*_{(N)})\ l = 4 \end{cases}, l = 1, 2, 3, 4. \tag{26}$$

7) **output:**
   a) The percentile bootstrap point estimate is given by
$$g_{boot-l} = \frac{1}{N}\sum_{i=1}^{N} g_l^{*(i)} \tag{27}$$
   b) Suppose that, $G(x) = P(\hat{g}_l^* \le x)$ be the CDF of $\hat{g}_l^*$, then $\hat{g}_l^* = G^{-1}(x)$. The $100(1-2\alpha)\%$ percentile bootstrap interval estimates are given by
$$\left(g^*_{boot-l}(\alpha),\ g^*_{boot-l}(1-\alpha)\right),\ l = 1, 2, 3, 4 \tag{28}$$

5. Bayesian estimation of the model parameters

Unlike frequentist methods, which focus on point estimates and confidence intervals. An alternative paradigm for statistical inference, which presents beliefs about parameters as prior knowledge, is known as the Bayesian approach. This approach produces posterior distributions as beliefs about parameters updating with observed data. Through this approach, we enable intuitive interpretations and the incorporation of prior information to yield full probabilistic descriptions of parameters. In this section, we explore the techniques for Bayesian estimation concepts applicable to the accelerated NXL distribution. Here, assume that the accelerated factor parameters θ has a non-informative prior distribution with a gamma distribution of the model parameter β given by

$$\Psi^*(\theta) \propto \theta^{-1}, \theta > 0, \tag{29}$$

and

$$\Psi^*(\beta) \propto \beta^{a-1}\, exp(-b\beta)\,, \beta > 0;\, a, b > 0. \tag{30}$$

The joint prior density of the parameters β and θ is given by

$$\Psi^*(\beta, \theta) \propto \theta^{-1}\beta^{a-1}\, exp(-b\beta). \tag{31}$$

The corresponding posterior density function is given by

$$\Psi(\beta, \theta|t) = \frac{\Psi^*(\beta,\theta)\times L(\beta,\theta|t)}{\int_1^\infty \int_0^\infty \Psi^*(\beta,\theta)\times L(\beta,\theta|t)d\beta d\theta}, \tag{32}$$

By using (11) and (31), the posterior distribution (32) is reduced to

$$\Psi(\beta, \theta|t) \propto \beta^{m_1+m_2+a-1}\theta^{m_2-1}\, exp\Bigg\{-b\beta + \sum_{i=1}^{m_1} log[\beta t_{i1} + 1] + \sum_{i=1}^{m_2} log[\beta t_{i2} + 1]$$
$$-\sum_{i=1}^{m_2} log\left[\frac{1}{2}\beta t_{i2} + 1\right] + \sum_{i=1}^{m_1} R_{i1}\, log\left[\frac{1}{2}\beta t_{i1} + 1\right] + \theta \sum_{i=1}^{m_2}(R_{i2} + 1)\, log\left[\frac{1}{2}\beta t_{i2} + 1\right]$$
$$-\beta \textstyle\sum_{i=1}^{m_1}(R_{i1} + 1)t_{i1} - \beta\theta \sum_{i=1}^{m_2}(R_{i2} + 1)t_{i2}\Bigg\}, \tag{33}$$

The Bayes estimate of any function under squared error loss function is the posterior mean. This often requires numerical methods such as numerical integration or MCMC method since closed-form solutions are typically unavailable.

Bayesian estimation under the MCMC approach

Gibbs sampling and Metropolis-within-Gibbs present an important scheme of MCMC methods used for Bayesian inference for complex models. A key advantage over MLE and bootstrap is that MCMC provides interval estimates and a full posterior distribution.

The conditional posterior distributions of the parameters β and θ can be formulated as

$$\Psi(\theta|\beta, t) \propto \theta^{m_2-1}\, exp\left\{\theta \textstyle\sum_{i=1}^{m_2}(R_{i2} + 1)\, log\left[\frac{1}{2}\beta t_{i2} + 1\right] - \beta\theta \sum_{i=1}^{m_2}(R_{i2} + 1)t_{i2}\right\}, \tag{34}$$

and

$$\Psi_2(\beta|\theta, t) \propto \beta^{m_1+m_2+a-1}\, exp\Bigg\{-b\beta + \sum_{i=1}^{m_1} log[\beta t_{i1} + 1] + \sum_{i=1}^{m_2} log[\beta t_{i2} + 1]$$
$$-\sum_{i=1}^{m_2} log\left[\frac{1}{2}\beta t_{i2} + 1\right] + \sum_{i=1}^{m_1} R_{i1}\, log\left[\frac{1}{2}\beta t_{i1} + 1\right] + \theta \sum_{i=1}^{m_2}(R_{i2} + 1)\, log\left[\frac{1}{2}\beta t_{i2} + 1\right]$$
$$-\beta \textstyle\sum_{i=1}^{m_1}(R_{i1} + 1)t_{i1} - \beta\theta \sum_{i=1}^{m_2}(R_{i2} + 1)t_{i2}\Bigg\}, \tag{35}$$

It's clearly from (34) the full conditional distribution of θ is a gamma distribution with shape parameter (m_2) and scale parameters $\left(\frac{1}{\beta \sum_{i=1}^{m_2}(R_{i2}+1)t_{i2} - \sum_{i=1}^{m_2}(R_{i2}+1)\, log\left[\frac{1}{2}\beta t_{i2}+1\right]}\right)$. Also, (43) shows that, $\frac{\partial^2\, ln\, \Psi_2(\beta|\theta,t)}{\partial \beta_1^2}$ are smaller than zero. Hence, the full conditional posterior distributions are log-concave; this property ensures a single maximum value and makes Metropolis-within-Gibbs a suitable algorithm for generating samples from the posterior distribution (43). To compute Bayesian estimates and construct credible intervals, the MCMC algorithm can be used to draw samples from the posterior distribution as follows

Algorithm 2: MCMC Algorithms:

1) **Input:** Objective functions are conditional distributions (35) and the normal distribution, initial guess value $\hat{\beta}$, $\hat{\theta}$ and iteration size M .
2) Put $s = 1$.
3) Generate $\beta^{(s)}$ from the gamma distribution (42).
4) Generate $\theta^{(s)}$ from (43) using MH algorithms with a normal proposal distribution with mean $\theta^{(s-1)}$ and standard error taken from the information matrix.

5) Compute the $\hat{S}(t)$ and $\hat{h}(t)$ values for given t .
6) Put $s = s + 1$.
7) Steps from 3 to 6 are repeated M times.
8) Arrange in ascending order all vectors $\beta_B = (\beta_B^1,\ \beta_B^2,\ \dots,\ \beta_B^M)$, $\theta_B = (\theta_B^1,\ \theta_B^2,\ \dots,\ \theta_B^M)$, $S_B = (S_B^1,\ S_B^2,\ \dots,\ S_B^M)$ and $h_B = (h_B^1,\ h_B^2,\ \dots,\ h_B^M)$ to obtain

$$\Omega_{B_l} = \begin{cases} (\beta_{B(1)}, \beta_{B(2)}, \dots, \beta_{B(M)}), l = 1 \\ (\theta_{B(1)}, \theta_{B(2)}, \dots, \theta_{B(M)})\ l = 2 \\ (S_{B(1)}, S_{B(2)}, \dots, S_{B(M)})\ l = 3 \\ (h_{B(1)}, h_{B(2)}, \dots, h_{B(M)})\ l = 4 \end{cases}, l = 1, 2, 3, 4. \tag{36}$$

9) output**:** The Bayes point estimate under squard loss function is given by

$$\Omega_{Bl} = \frac{1}{M - M^*} \sum_{i=M^*+1}^{M} \Omega_{Bl}^i, \tag{37}$$

where M^* is brun-in. Suppose that, $G(x) = P(\Omega_{Bl} \le x)$ be the CDF of Ω_{Bl}, then $\Omega_{Bl} = G^{-1}(x)$. The $100(1 - 2\alpha)\%$ credible interval estimates are given by

$$(\Omega_{Bl}(\alpha), \Omega_{Bl}(1 - \alpha)), l = 1, 2, 3, 4. \tag{38}$$

6. Simulation Studies

To evaluate the performance of the proposed estimation methods, a comprehensive simulation study is conducted. The study is carried out in Mathematica (ver. 12.0). Datasets are generated from the NXL distribution under various scenarios. To assess the accuracy of the point estimates, the average (AVG) and mean square error (MSE) are computed for the model parameter and the accelerated factor. MSE is selected for assessing and comparing different estimators rather than the Diebold-Mariano test because of its computational simplicity, widespread interpretability, and direct penalization of large errors. The Diebold-Mariano test requires a sufficiently large sample size to achieve reliable asymptotic properties that are not consistent with the censoring scheme. Interval estimators are evaluated using the coverage percentage (CP) and mean interval length (MIL). The results provide insights into the relative performance of the methods across different sample sizes, schemes, and parameter settings. The simulation results are based on 1000 samples generated from the proposed model. The simulation step is described as follows.

Algorithm 3: (Monte Carlo study)

1) **Input:** n, n_j, m_j, η, and the CSs r_j, $j = 1,2$.
2) For given β and θ generate an accelerated adaptive type-II PHC sample, $t = \{t_{1j},\ t_{2j},\ \dots, t_{k_j j}, t_{(k_j+1)j}, \dots, t_{m_j j}\}$.
3) Reported from the data k_j , and the corrected censoring schemes R_j .
4) Compute for each random sample the estimated values ML, bootstrap, and Bayes estimation for the parameter, and the corresponding parameters of life.
5) The interval estimates are computed for different approaches.
6) Steps from 2 to 5 are repeated 1000 times.
7) **Output:** The AVG, MSE, CP, and MILs for each sample.

In our study, without loss of generality, choice $n_1 = n_2 = n$ and $m_1 = m_2 = m$ for CS, we adopt four schemes

1) I_1 scheme : $R_i = 0$ for all $i < m$, $R_m = n - m$.
2) I_2 scheme : $\begin{cases} R_i = 1, when\ \frac{m+1}{2} \le i \le \frac{2n-m+1}{2} \text{ if m is odd.} \\ R_i = 1, when\ \frac{m}{2} \le i \le \frac{2n-m}{2} \text{ if m is even.} \\ R_i = 0, OW \end{cases}$
3) I_3 scheme : $R_i = 1$, if $m = \frac{n}{2}$.
4) I_4 scheme : $R_1 = n - m$, $R_i = 0$ for $i \ne 1$

Also, we chose the one set of parameters { β , θ }= {0.2, 1.5} to generate 1000 different sets of accelerated adaptive type-II PHC samples. For the Bayesian approach, adopt non-informative prior information {a,b}={0.0001, 0.0001} and informative prior information { a , b }={1.0, 3.0}, and run chain 11000, removing the first 1000 iterations as burn-in.

7. Data Analysis

In this section, we present analyses of real and simulated datasets to illustrate the practical application of the proposed methods. The results of this section have shown the flexibility and usefulness of the NE-LD in handling real-world phenomena.

Table 1. AVG and (MSEs) for the estimate of the parameters { β , θ } = {0.2, 1.5}.

(N,η)	(n,m)	CS	MLE		Boot		Bayes (P^0)		Bayes (P^1)	
			β	θ	β	θ	β	θ	β	θ
(60,5)	(30,15)	I_1	0.345	1.845	0.381	1.884	0.332	1.818	0.278	1.645
			0.0989	0.2181	0.1029	0.2270	0.0978	0.2169	0.0785	0.1878
		I_2	0.324	1.819	0.355	1.867	0.309	1.801	0.255	1.625
			0.0989	0.2181	0.1029	0.2270	0.0978	0.2169	0.0785	0.1878
		I_3	0.317	1.817	0.357	1.858	0.304	1.803	0.251	1.619
			0.0974	0.2171	0.1011	0.2255	0.0968	0.2149	0.0781	0.1868
		I_4	0.289	1.782	0.331	1.829	0.277	1.775	0.219	1.589
			0.0931	0.2125	0.0981	0.2211	0.0935	0.2108	0.0751	0.1821
	(30,25)	I_1	0.318	1.818	0.351	1.859	0.308	1.791	0.257	1.627
			0.0930	0.2128	0.0982	0.2214	0.0924	0.2122	0.0732	0.1821
		I_2	0.292	1.790	0.331	1.841	0.285	1.771	0.247	1.607
			0.0927	0.2129	0.0885	0.2215	0.0921	0.2111	0.0730	0.1822
		I_3	0.289	1.791	0.318	1.829	0.287	1.783	0.238	1.607
			0.0918	0.2115	0.0879	0.2201	0.0911	0.2102	0.0737	0.1808
		I_3	0.263	1.757	0.307	1.802	0.255	1.751	0.210	1.569
			0.0887	0.2081	0.0922	0.2164	0.0891	0.250	0.0696	0.1781
(100,5)	(50,25)	I_1	0.301	1.795	0.333	1.844	0.300	1.777	0.235	1.614
			0.0919	0.2120	0.0971	0.2202	0.0904	0.2108	0.0718	0.1811
		I_2	0.274	1.781	0.319	1.827	0.271	1.754	0.239	1.589
			0.0912	0.2114	0.0869	0.2200	0.0904	0.2097	0.0718	0.1802
		I_3	0.270	1.781	0.307	1.811	0.271	1.767	0.220	1.600
			0.0905	0.2102	0.0863	0.2189	0.0902	0.2087	0.0730	0.1802
		I_4	0.248	1.734	0.295	1.791	0.237	1.718	0.2010	1.560
			0.0872	0.2066	0.0909	0.2148	0.0881	0.234	0.0682	0.1769
	(50,45)	I_1	0.257	1.751	0.289	1.800	0.261	1.725	0.176	1.555
			0.0820	0.2015	0.0899	0.2098	0.0801	0.207	0.0631	0.1701
		I_2	0.241	1.751	0.289	1.800	0.235	1.719	0.204	1.551
			0.0800	0.2114	0.0780	0.2097	0.0850	0.2002	0.0624	0.1691
		I_3	0.241	1.744	0.285	1.755	0.244	1.739	0.175	1.561
			0.0782	0.201	0.0751	0.2028	0.0814	0.2001	0.0615	0.1714

		I_4	0.2014	1.690	0.266	1.757	0.202	1.64	0.198	1.513
			0.0782	0.1045	0.0801	0.2014	0.0780	0.122	0.0602	0.1612

Table 2. AVG and (MSEs) for the estimate of the $S(t)$ and $h(t)$ $\{ \beta , \theta, t \} = \{0.2, 1.5, 1.0\}$

(N, η)	(n, m)	CS	MLE		Boot		Bayes (P^0)		Bayes (P^1)	
			$S(1.0)$	$h(1.0)$	$S(1.0)$	$h(1.0)$	$S(1.0)$	$h(1.0)$	$S(1.0)$	$h(1.0)$
(60,5)	(30,15)	I_1	0.8782	0.5188	0.8692	0.5100	0.8795	0.5200	0.8882	0.5297
			0.0879	0.0754	0.0930	0.0820	0.0865	0.0738	0.0780	0.0681
		I_2	0.8821	0.5215	0.8722	0.5131	0.8825	0.5242	0.8910	0.5328
			0.0861	0.0731	0.0914	0.0802	0.0847	0.0721	0.0758	0.0665
		I_3	0.8845	0.5231	0.8747	0.5155	0.8861	0.5285	0.8935	0.5347
			0.0852	0.0728	0.0901	0.0793	0.0835	0.0709	0.0750	0.0652
		I_4	0.8891	0.5277	0.8780	0.5198	0.8910	0.5311	0.8968	0.5381
			0.0824	0.0703	0.0878	0.0771	0.0807	0.0691	0.0738	0.0650
	(30,25)	I_1	0.8830	0.5237	0.8751	0.5145	0.8815	0.5261	0.8920	0.5335
			0.0858	0.0741	0.0911	0.0803	0.0838	0.0721	0.0754	0.0662
		I_2	0.8881	0.5271	0.8769	0.5177	0.8870	0.5282	0.8971	0.5360
			0.0839	0.0708	0.0900	0.0781	0.0831	0.0701	0.0734	0.0641
		I_3	0.8892	0.5297	0.8799	0.5201	0.8901	0.5330	0.8982	0.5381
			0.0815	0.0709	0.0881	0.0779	0.0817	0.0691	0.0728	0.0619
		I_3	0.8941	0.5310	0.8831	0.5238	0.8971	0.5365	0.9050	0.5420
			0.0803	0.0687	0.0858	0.0744	0.0782	0.0675	0.0738	0.0632
(100,5)	(50,25)	I_1	0.8835	0.5244	0.8762	0.5152	0.8823	0.5264	0.8928	0.5341
			0.0841	0.0728	0.0900	0.0789	0.0821	0.0704	0.0750	0.0649
		I_2	0.8893	0.5281	0.8781	0.5180	0.8881	0.5288	0.8979	0.5373
			0.0830	0.0695	0.0891	0.0769	0.0817	0.0692	0.0730	0.0629
		I_3	0.8905	0.5307	0.8810	0.5214	0.8915	0.5341	0.8989	0.5392
			0.0804	0.0701	0.0869	0.0765	0.0808	0.0679	0.0720	0.0608
		I_4	0.8954	0.5318	0.8845	0.5250	0.8979	0.5372	0.9064	0.5434
			0.0791	0.0680	0.0849	0.0737	0.0771	0.0670	0.0729	0.0618
	(50,45)	I_1	0.8861	0.5270	0.8788	0.5188	0.8861	0.5299	0.8963	0.5352
			0.0815	0.0702	0.0881	0.0751	0.0790	0.0677	0.0714	0.0622
		I_2	0.8921	0.5309	0.8815	0.5205	0.8920	0.5318	0.8992	0.5398
			0.0803	0.0669	0.0864	0.0742	0.0800	0.0664	0.0705	0.0592
		I_3	0.8941	0.5342	0.8838	0.5237	0.8949	0.5369	0.8020	0.5417
			0.0775	0.0681	0.0842	0.0741	0.0791	0.0662	0.0700	0.0589
		I_4	0.8976	0.5341	0.8869	0.5277	0.8998	0.5395	0.9088	0.5459
			0.0771	0.0655	0.0831	0.0719	0.0744	0.0645	0.0699	0.0589

Table 3. The CP and AIL model parameters

(N,η)	(n,m)	CS	MLE		Boot		Bayes (P^0)		Bayes (P^1)	
			β	θ	β	θ	β	θ	β	θ
(60,5)	(30,15)	I_1	0.90	0.89	0.90	0.91	0.91	0.90	0.91	0.92
			0.456	3.453	0.522	3.574	0.422	3.432	0.389	3.120
		I_2	0.91	0.92	0.93	0.92	0.94	0.90	0.93	0.94
			0.442	3.438	0.509	3.563	0.408	3.421	0.377	3.103
		I_3	0.90	0.94	0.94	0.96	0.94	0.93	0.96	0.93
			0.435	3.430	0.503	3.554	0.397	3.413	0.371	3.095
		I_4	0.93	0.92	0.94	0.92	0.94	0.93	0.94	0.95
			0.414	3.413	0.489	3.541	0.379	3.395	0.355	3.077
	(30,25)	I_1	0.91	0.90	0.92	0.91	0.93	0.91	0.91	0.91
			0.431	3.429	0.501	3.561	0.404	3.415	0.371	3.103
		I_2	0.92	0.93	0.93	0.94	0.94	0.93	0.93	0.95
			0.425	3.421	0.491	3.549	0.392	3.401	0.354	3.087
		I_3	0.92	0.94	0.93	0.95	0.93	0.94	0.93	0.94
			0.414	3.411	0.487	3.540	0.387	3.392	0.354	3.087
		I_3	0.92	0.92	0.91	0.92	0.92	0.92	0.92	0.94
			0.400	3.392	0.471	3.527	0.370	3.382	0.337	3.062
(100,5)	(50,25)	I_1	0.92	0.97	0.91	0.91	0.94	0.93	0.92	0.93
			0.428	3.424	0.497	3.554	0.400	3.407	0.366	3.097
		I_2	0.91	0.94	0.94	0.94	0.93	0.93	0.92	0.94
			0.417	3.412	0.486	3.542	0.387	3.395	0.350	3.080
		I_3	0.91	0.91	0.93	0.94	0.93	0.93	0.92	0.92
			0.404	3.403	0.481	3.532	0.382	3.388	0.350	3.079
		I_4	0.95	0.92	0.95	0.92	0.92	0.95	0.93	0.96
			0.392	3.385	0.465	3.523	0.363	3.380	0.331	3.051
	(50,45)	I_1	0.91	0.92	0.91	0.93	0.92	0.93	0.94	0.91
			0.401	3.397	0.480	3.532	0.374	3.282	0.341	3.065
		I_2	0.92	0.93	0.94	0.93	0.93	0.93	0.96	0.95
			0.391	3.380	0.459	3.517	0.365	3.374	0.324	3.055
		I_3	0.93	0.91	0.92	0.92	0.93	0.93	0.94	0.96
			0.387	3.369	0.449	3.500	0.347	3.344	0.317	3.054
		I_4	0.93	0.94	0.95	0.94	0.92	0.94	0.93	0.92
			0.366	3.364	0.439	3.500	0.334	3.351	0.297	3.019

Table 4. The CP and AIL parameters of life $S(1.0)$, $h(1.0)$.

(N,η)	(n,m)	CS	MLE		Boot		Bayes (P^0)		Bayes (P^1)	
			β	θ	β	θ	β	θ	β	θ
(60,5)	(30,15)	I_1	0.92	0.88	0.92	0.90	0.92	0.92	0.93	0.91
			1.845	1.254	1.932	1.341	1.822	1.239	1.647	1.125
		I_2	0.91	0.90	0.93	0.92	0.92	0.94	0.92	0.93
			1.817	1.228	1.914	1.325	1.804	1.217	1.631	1.108
		I_3	0.93	0.93	0.91	0.94	0.92	0.93	0.93	0.94
			1.805	1.217	1.901	1.316	1.797	1.205	1.622	1.100
		I_4	0.92	0.93	0.92	0.93	0.92	0.95	0.93	0.95
			1.775	1.200	1.881	1.292	1.771	1.184	1.594	1.74
	(30,25)	I_1	0.91	0.92	0.92	0.93	0.92	0.93	0.93	0.94
			1.819	1.224	1.907	1.315	1.800	1.211	1.618	1.101
		I_2	0.93	0.92	0.93	0.94	0.92	0.93	0.92	0.96
			1.792	1.203	1.891	1.300	1.782	1.189	1.605	1.087
		I_3	0.94	0.93	0.92	0.94	0.92	0.93	0.94	0.94
			1.777	1.200	1.874	1.291	1.766	1.184	1.594	1.070
		I_3	0.91	0.93	0.94	0.93	0.94	0.93	0.93	0.94
			1.755	1.171	1.853	1.271	1.747	1.156	1.571	1.482
(100,5)	(50,25)	I_1	0.93	0.92	0.93	0.93	0.94	0.93	0.91	0.96
			1.804	1.213	1.897	1.303	1.792	1.202	1.603	1.091
		I_2	0.92	0.92	0.91	0.94	0.92	0.94	0.92	0.94
			1.781	1.187	1.883	1.289	1.770	1.180	1.593	1.076
		I_3	0.91	0.94	0.92	0.94	0.93	0.94	0.96	0.93
			1.765	1.187	1.865	1.279	1.751	1.172	1.580	1.060
		I_4	0.93	0.93	0.92	0.93	0.95	0.93	0.94	0.93
			1.741	1.159	1.844	1.258	1.735	1.150	1.563	1.471
	(50,45)	I_1	0.92	0.91	0.93	0.93	0.94	0.92	0.91	0.93
			1.765	1.181	1.855	1.272	1.766	1.169	1.567	1.044
		I_2	0.93	0.92	0.93	0.94	0.94	0.94	0.96	0.93
			1.751	1.144	1.839	1.251	1.741	1.144	1.551	1.049
		I_3	0.93	0.94	0.92	0.92	0.93	0.94	0.92	0.95
			1.729	1.159	1.833	1.245	1.717	1.154	1.561	1.033
		I_4	0.94	0.92	0.92	0.94	0.95	0.94	0.92	0.94
			1.707	1.122	1.813	1.224	1.703	1.117	1.528	1.448

7.1. Example 1 (real-life application)

In this example, two real datasets are used to show the real-life applicability of NE-LD under accelerated life testing. The first real data set was discussed by Meeker et al. [Meeker et al.2022] and fit on the NE-LD by Khodja et al. [Khodja et al.2023] to present time-to-failure and running times for larger system field-tracking research on a sample of devices. For computation simplicity, divide the data by 10 to give 0.2, 1.0, 1.3, 2.3, 2.3, 2.8, 3.0, 6.5, 8.0, 8.8, 10.6, 14.3, 14.7, 17.3, 18.1, 21.2, 24.5, 24.7, 26.1, 26.6, 27.5, 29.3, 30.0, 30.0, 30.0, 30.0, 30.0, 30.0, 30.0, 30.0. The second data set was taken under different temperatures to indicate the failure times of eight components Murthy et al.2004 to give after divide the data by 10 as, 0.2998, 0.5016, 0.861, 1.1741, 1.4712, 1.5628, 2.304, 2.7851, 3.2644, 3.7843, 3.805, 4.8226, 5.4535, 5.5047, 5.8928, 6.1979, 6.3391, 6.5521, 10.518, 10.55, 11.302, 11.46, 12.04, 13.85. These two datasets were fitted to the NEL distribution in Khodja et al. [Khodja et al.2023]. Let, $n_1 = m_1 = 30$, $n_2 = m_{21} = 24$, the ideal test time η =20 and censoring scheme are given by $\boldsymbol{r_1}$ ={ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 } and $\boldsymbol{r_2}$={ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 }. We obtain the complete data as a special case of accelerated adaptive type-II PHCS. A non-informative prior is used in Bayes approach. With chain iteration, 11000 steps are run with respect to the MCMC method, and the first 1000 iterations are considered burn-in. The results of ML, bootstrap, and Bayes point and interval estimates are reported in Table 5. Convergence diagnostics of MCMC method are shown in Figures 6-9.

Table 5. The point and corresponding 95% interval estimate of ML, bootstraps and Bayes approach.

Parameter	(.) ML	(.) Boot	(.) Bayes	ACI	95% Boot-p	95% Bayes
β	0.0861	0.0924	0.0860	(0.0637, 0.10857)	(0.0453, 0.1148)	(0.0651, 0.1088)
β	3.6871	3.8741	3.8042	(2.0615, 5.3127)	(2.3111, 5.6798)	(2.3549, 5.6578)
$S(1.2)$	0.9484	0.9321	0.9485	-	(0.9247, 0.9741)	(0.9349, 0.961)
$h(1.2)$	0.0452	0.0541	0.0451	-	(0.0300, 0.0745)	(0.0338, 0.0578)

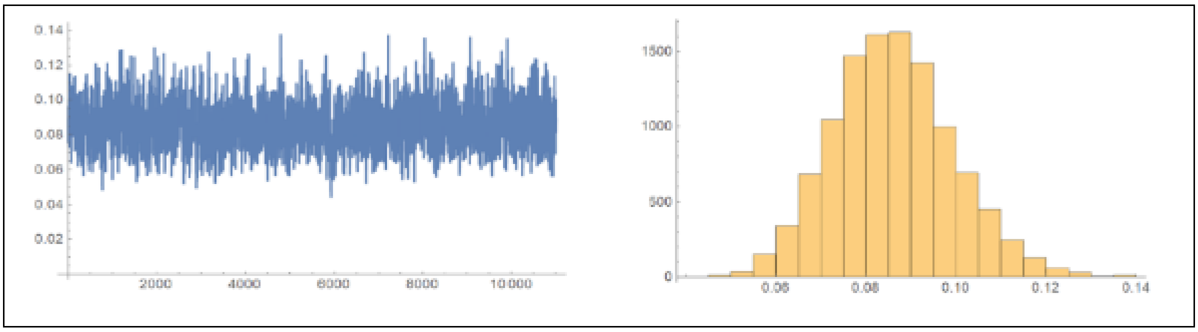

Fig. 2. Simulation number and corresponding histogram of β generated by the MCMC method.

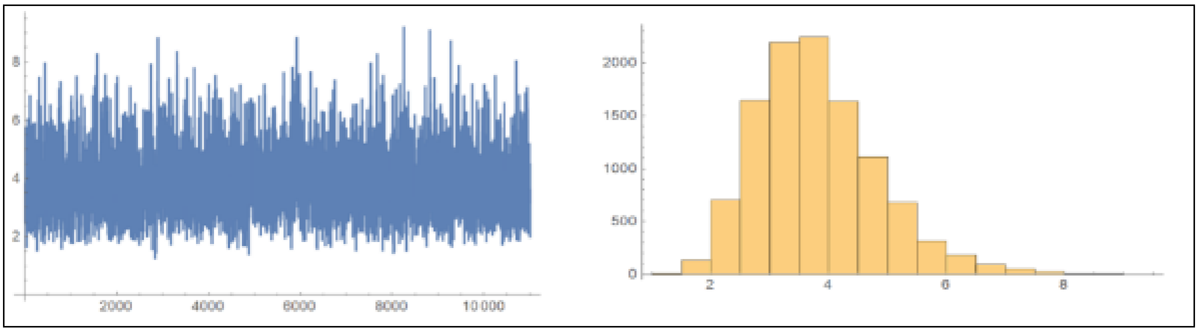

Fig. 3. Simulation number and corresponding histogram of θ generated by the MCMC method.

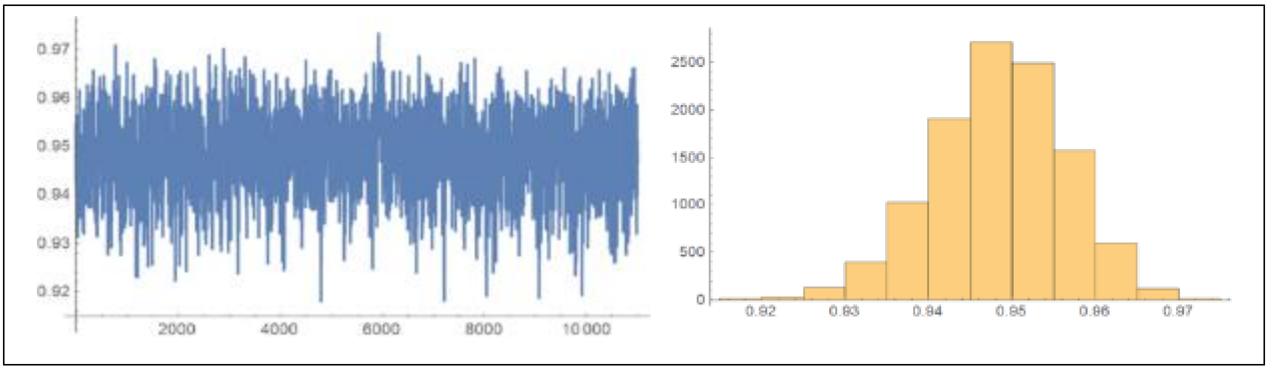

Fig. 4. Simulation number and corresponding histogram of $S(t)$ generated by the MCMC method.

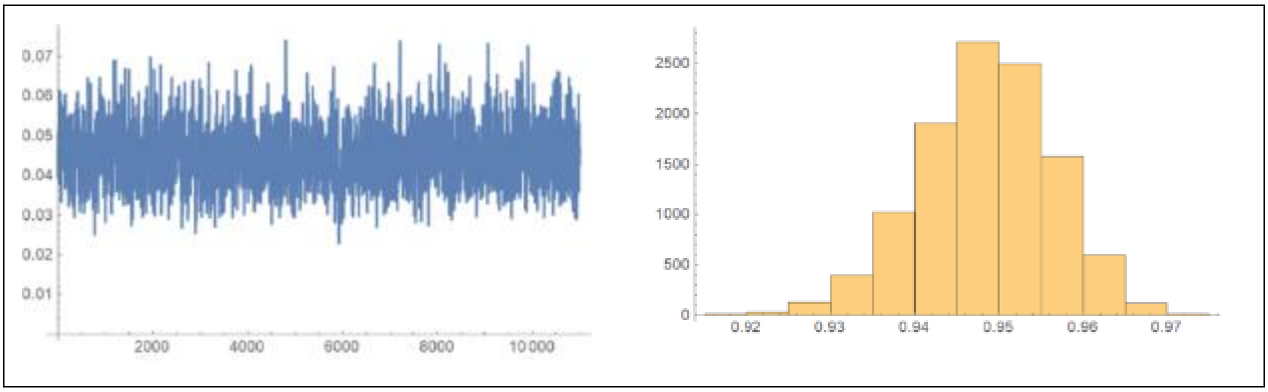

Fig. 5. Simulation number and corresponding histogram of $h(t)$ generated by the MCMC method.

7.2. Numerical example

A simulated accelerated adaptive PHC generated from the NEL distribution is used to illustrate the application of the proposed methods. Under consideration, $\beta = 0.8$, $\theta = 2.0$, $\eta = 0.9$, $n_1 = 50$, $m_1 = 20$, $n_2 = 50$, $m_2 = 30$, $r_1 = \{$ 5, 0, 0, 5, 0, 0, 3, 0, 0, 0, 5, 2, 2, 2, 1, 1, 1, 1, 1, 1 $\}$ and $r_2 = \{$3, 0, 0, 0, 2, 0, 0, 0, 2, 0, 0, 0, 2, 0, 0, 0, 2, 0, 0, 3, 0, 1, 0, 1, 0, 1, 0, 0, 2, 1$\}$. By using inverse transormation and (9) and the algorithms given by Ng et al. [Ng et al.2010] the simulated data given by { 0.00289082, 0.0706965, 0.108449, 0.124012, 0.17445, 0.246222, 0.252854, 0.30069, 0.471841, 0.532262, 0.538561, 0.653538, 1.00575, 1.23743, 1.32057, 1.86324, 1.89976, 1.96496, 2.14788, 2.15169} under use condition and {0.0110125, 0.0577813, 0.0787624, 0.159907, 0.161399, 0.193324, 0.19752, 0.236194, 0.337404, 0.348553, 0.468478, 0.51857, 0.52085, 0.533235, 0.595526, 0.607602, 0.630872, 0.639245, 0.663427, 0.670323, 0.703967, 0.889473, 1.19654, 1.39471, 1.54981, 1.61894, 1.64854, 1.70899, 1.78754, 1.85287} under accelerated condition. For the generated data, the corrected censoring SCs is given by $R_1 = \{$ 5, 0, 0, 5, 0, 0, 3, 0, 0, 0, 5, 2, 0, 0, 0, 0, 0, 0, 0, 10 $\}$ and $R_2 = \{$ 3, 0, 0, 0, 2, 0, 0, 0, 2, 0, 0, 0, 2, 0, 0, 0, 2, 0, 0, 3, 0, 1, 0, 0, 0, 0, 0, 0, 0, 5 $\}$ and $k_1 = 12$, $k_2 = 22$. The results of the point estimate are reported in Table 7, and from the interval estimation in Table 6. Informative priors with a =2.0 and b =3.0 in Bayesian analysis was performed using MCMC simulation with running 11000 iterations (burn-in: 1000). Convergence diagnostics are shown in Figures 2-5.

Table 6. The point and corresponding 95% interval estimate of ML, bootstraps, and the Bayes approach.

Parameter	(.) ML	(.) Boot	(.) Bayes	ACI	95% Boot-p	95% Bayes
$\beta = 0.8$	0.7413	0.8425	0.8515	(0.499, 0.9837)	(0.4141, 1.2412)	(0.6392, 1.0822)
$\beta = 2.0$	1.8442	1.8784	1.3954	(0.9756, 2.7128)	(0.9666, 2.7781)	(0.908, 2.1014)
$S(1.2) = 0.5667$	0.5936	0.6074	0.5463	-	(0.4214, 0.7452)	(0.4501, 0.6425)
$h(1.2) = 0.5297$	0.4848	0.4758	0.5713	-	(0.4332, 0.8542)	(0.4082, 0.7541)

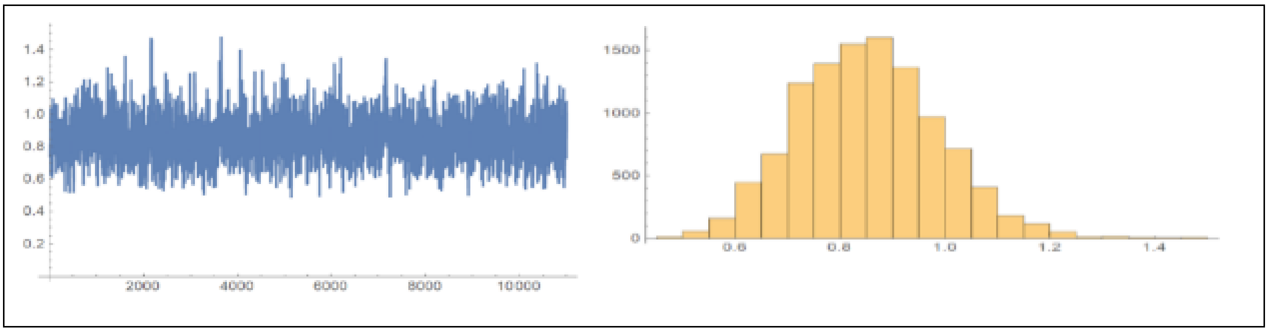

Fig. 6. Simulation number and corresponding histogram of β generated by the MCMC method.

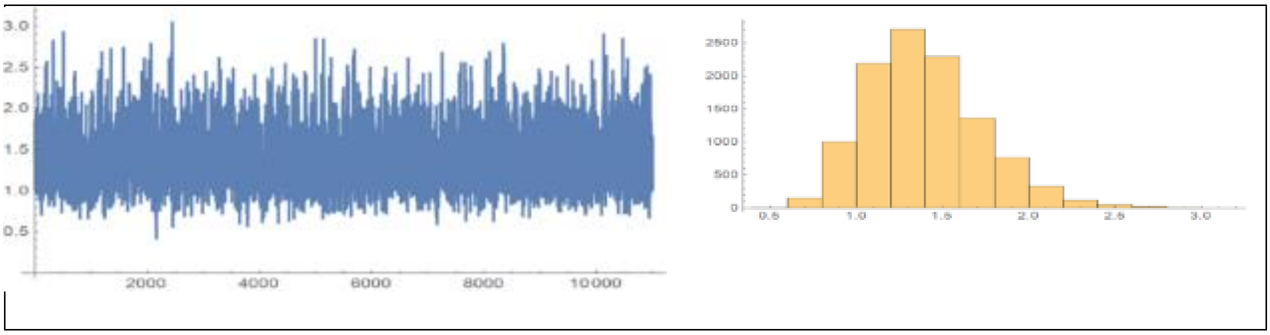

Fig. 7. Simulation number and corresponding histogram of θ generated by the MCMC method.

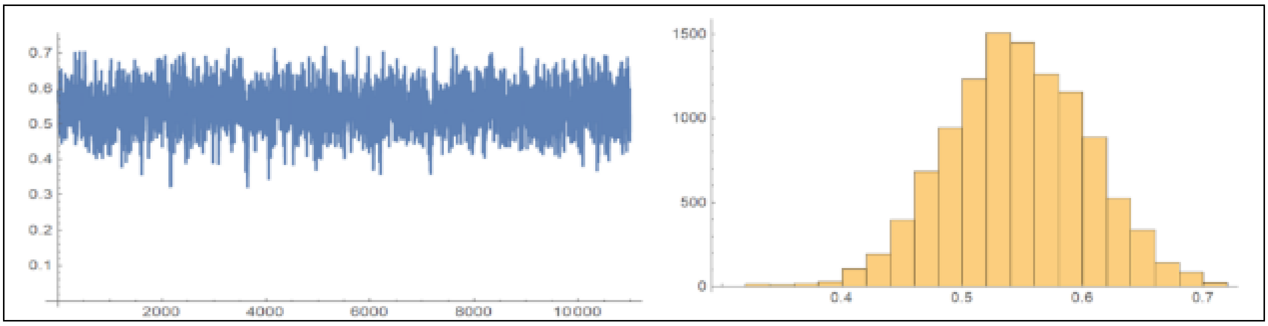

Fig. 8. Simulation number and corresponding histogram of $S(t)$ generated by the MCMC method.

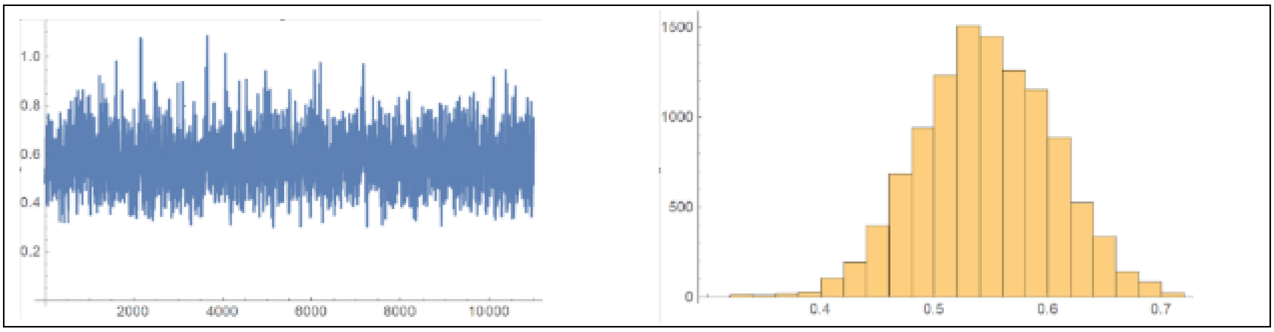

Fig. 9. Simulation number and corresponding histogram of h(t) generated by the MCMC method.

8. Numerical discussion

A Monte Carlo simulation study was conducted in Section 6 to evaluate the performance of the proposed estimation methods. The results taken from Tables 1 to 4 show. Across all scenarios, satisfactory performance of classical and Bayesian estimators. Satisfying the validated consistency and accuracy of estimators. Relative strengths and weaknesses of methods highlighted.

a. For reliability studies, the adaptive type-II PHCS is practical for life-testing experiments.
b. When reliable items with failure time are present, the accelerated models are essential for accurate analysis.
c. ML and non-informative Bayes estimates produce similar results.
d. Informative Bayes estimates are the best choice among estimation methods.
e. Bootstrap CIs and asymptotic CIs perform as non-informative symmetric credible intervals.
f. MSE decreases as the proportion (m/n) increases, improving estimation.
g. Mean interval length (MIL) decreases as (m/n) increases.
h. Coverage percentage approaches the nominal level (1-2 α)% as (m/n) increases, indicating better interval estimate reliability. accuracy.

From the data Analysis section, real and simulated datasets were analyzed using the NE-XD, and the results show. NE-XD fits data adequately as given in Khodja et al. [Khodja et al.2023]. Maximum likelihood, bootstrap, and Bayesian estimates are comparable. Practical applicability of the proposed methods is demonstrated.

9. Conclusions and Some Comments

This study proposed the NE-LD for modeling lifetime data. Also, we adopt the maximum likelihood, bootstrap, and Bayesian estimation methods to develop. We discuss the improved estimation of the model parameters and the parameters of life (reliability and failure rate functions), which is a tool used to contribute to broader societal and environmental benefits as well as enhance the operational safety and dependability of electronic systems. Hence, reducing the likelihood of unexpected failures in safety-critical applications requires the accurate prediction of device reliability. In addition, better reliability assessment supports optimized maintenance and longer product lifecycles. A simulation study and real data analyses demonstrated the model's flexibility and the methods' performance. Results show satisfactory estimation accuracy and practical applicability, highlighting the NE-LD potential for reliability and survival analysis. The results show that the outperformance of informative Bayesian estimators in point and interval estimates over maximum likelihood and bootstrap estimates.

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